Note

A Note to Kazuo Muroi, "Inheritance Problems in the Susa Mathematical Text No. 26"[1]

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In a recent paper,[2] Kazuo Muroi presents an ingenious and convincing interpretation of a text from late Old Babylonian Susa which so far has resisted deciphering, in part because the cursive script was wrongly read on some important points in the original edition, in part because errors in the text itself have obstructed the reconstruction of damaged passages.

As always suspected, the problems of the text deal with partitions of (practically rectangular) trapezoidal fields; they are only "inheritance problems" in the sense that two of the four problems refer to the partition as a division between two "brothers".

According to Muroi, the "first problem is so badly damaged that we cannot understand the mathematical purport except that it deals with a trapezoid". However, performance of a calculation unambiguously initiated in line 2 allows us to go somewhat beyond this admission of defeat, completing Muroi's transliteration as follows:

*. [... ... ...]
1. *šà* ... [... ...] ... [...]
2. 2,10 sag a[n.na] ugu 30 s[ag ki.ta 1,40 dirig]
3. igi 3,45 uš dug$_8$-*ma* 16 [*a-na* 1,40 íl]
4. 26,40 a.rá 2 53,20 *a-n*[*a X* íl *Y*]
5. *ta-ta-a-ar* 2,10 sag an.n[a gu$_7$.gu$_7$-*ma* 4,41,40]
6. 4 *i-na* 4,41,40 kud-*ma* [41,40]

with translation, omitting the square brackets

*. ... ...
1. that of ... ...
2. 2,10, the upper width, exceeds 30, the lower width, by 1,40.
3. Detach the reciprocal of 3,45, the length, (it is) 0;0,16, raise to 1,40,
4. 0;26,40; times 2, 0;53,20, raise to X, (it is) *Y*.

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[1] The note elaborates a point that could only be sketched in my abstract of the paper in *Mathematical Reviews* **2002c**:01005.

[2] "Inheritance Problems in the Susa Mathematical Text No. 26" *Historia Scientiarum*, second series 10:3 (2001), 226–234.

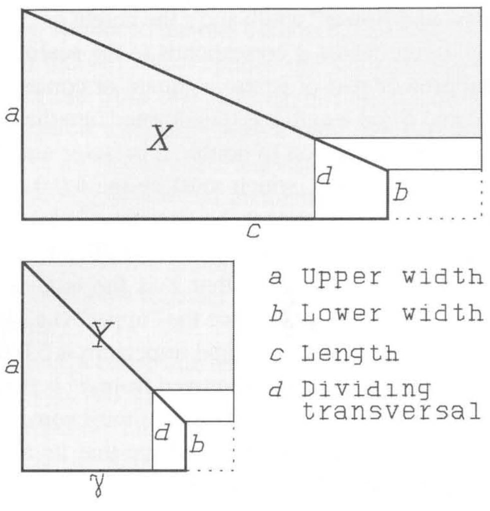

Figure 1.

5. You return. 2,10, the upper width, make hold, (it is) 4,41,40.
6. 4,0,0 from 4,41,40 cut off, (it is) 41,40.

Here I have followed the principles of Muroi's translation with these exceptions:[3]

—A reciprocal is "detached" (du_8, a word sign for Akkadian *paṭārum*), in agreement with the general semantics of this verb; the meaning is probably that finding $^1/_n$ is seen as detaching one part from a bundle containing n parts;

—íl (word sign for *našûm*) is translated "to raise" in order to keep this particular multiplication (corresponding to an operation of proportionality) apart from the others – for instance from a.rá, "steps of", here translated "times", used about the arithmetical multiplication of number by number:

—for *târum* I use the literal translation "to return"; the meaning remains that of demarcating the transition to a new section of the procedure.

—$gu_7.gu_7$ (written kú.kú by Muroi, word sign for *šutakūlum*) I translate "to make hold", in agreement with the use of the verb to designate the process in which two lines are caused to contain a rectangle (if as here only one is mentioned, a square).[4]

—kud is interpreted as a word sign for *ḫarāṣum*, "to cut off", which agrees better with what else we know about the logographic function of that sign than Muroi's reading *nasāḫum*. The technical meaning is unchanged, a "subtraction by removal".

Lines 2–4 find the number $\varphi = {}^{a-b}/_c = 0;26,40 = {}^4/_9$, which shows the text to aim at application of a technique that is used routinely in the Old Babylonian mathematical corpus

[3] For detailed discussion of the terms and their translations, see my *Lengths, Widths, Surfaces* (New York: Springer, 2002), pp. 20–30.

[4] The alternative derivation of the term from *akālum*, "to eat" proposed by Neugebauer and accepted since then by most workers (including Muroi but excluding Thureau-Dangin) in spite of the semantic enigma it presents us with, is excluded by the interchangeability of the relative phrase *ša tu-ušta-ki-lu* and the noun *takīltum*, indubitably derived from the verb *kullum*, "to hold".

(a and b being the "upper" and "lower" width and c the height or "length" of the trapezium, see Figure 1). "Raising" to the factor φ corresponds to the scaling in one dimension that transforms a rectangular area or part of it into a square or corresponding part—here, the trapezium with widths a and b and length c is transformed into the half of a square gnomon (cut along the diagonal). Line 4 goes on to double this factor and then applies it to some number X, finding another number Y, which must be the 4,0,0 appearing in line 6 (this number can come from nowhere else except the statement, while the number Y found in line 4 can have no other use). Lines 5–6 find a^2 = 4,41,40, subtracts 4,0,0 from this and finds a remainder 41,40 = 50^2. This shows that Y is the total shaded area 4,0,0 in the lower part of the diagram, and X = 4,30,0 hence the "upper" (i.e., left) part of the original trapezium, whereas the whole area of the original trapezium is 5,0,0. As can be seen from the diagram, the 50^2 that remains when Y is removed from a^2 is the square on the dividing transversal d, which means that d is 50. The statement must somehow have told the upper part of the trapezium to have the area 4,30,0 (perhaps that its area is $^9/_{10}$ of the whole trapezium), and it is likely to have asked for the length d, even though only the square on d is actually found.

Problem 2 is a bisection, in which b is told to be 30 and the bisecting transversal d to be 50. It first finds $(d - b) \cdot (d + b) = 26,40$ and then doubles this number with result $2 \cdot (d - b) \cdot (d + b) = 53,20$; next it calculates d^2 = 41,40. Finally it finds a as the approximate square root of 1,8,20. The number 1,8,20 can be either 53,20+15,0 (that is, $2 \cdot (d+b) \cdot (d-b) + b^2$) or 26,40+41,40 (that is, $(d+b) \cdot (d-b) + d^2$); both possibilities are pointed at by Muroi; the two numbers that are calculated in the text obviously confuse the two formulae. The adjacent diagram explains why both are correct, and why they are easily confused; since only widths and the transversal enter the problem, we may as well imagine the length to be equal to a–b, in which case the trapezium is already one half of a square gnomon. Then the area of the inner half of the trapezium is $^{(d+b)\cdot(d-b)}/_2$, and the whole shaded zone thus $(d + b) \cdot (d - b)$. But since d bisects the trapezium, this must also be the area of the black zone, and the whole square a^2 is hence $2 \cdot (d + b) \cdot (d - b) + b^2$, but also $(d + b) \cdot (d) + d^2$. Mixing the two ideas is only too easy.

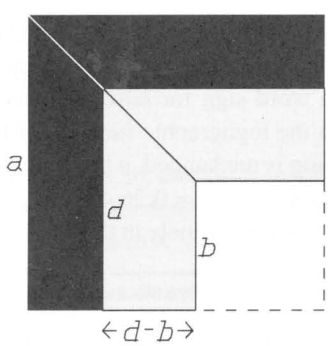

Figure 2.

A trick similar to the one used in #2 is found in the text YBC 4675, another bisection problem; YBC 4675 also applies the same scaling to square shape as #1 of the present text.[5] Like our Susa text (and against normal Babylonian habits) YBC 4675 also fails to make explicit a division which only changes the implicit order of magnitude or the dimension of a number (there a division by 1,0,0; here, in obv. II 6, by 1.)[6] Strikingly, the same sin (ending up there in mathematical nonsense) is committed in the somewhat later text AO 17264, obv. 4–10, yet another division of a trapezoidal field into parallel strips, also making use of the scaling to square shape.[7] For obscure reasons, the problem type appears to have gone together with this dubious way to short-circuit the sexagesimal place value system.

Once this peculiar link between the three texts has been noticed, one further similarity catches the eye. AO 17264 contains a pseudo-solution to a problem whose correct solution was far beyond the range of methods known at the time: the division of a trapezium into six parallel strips that are pairwise equal, that is, into three rationally bisectable trapezia. In all likelihood, the numerical correctness of the fake solution is due to the fact that the problem was constructed backwards from a known solution, obtained from the gluing-together of known bisectable trapezia, all easily found by inspection of the table of squares or by experimental squaring.[8]

As pointed out by Muroi, problem 4 of the Susa text ends paradoxically by giving an exact final solution apparently based on an intermediate approximate step (though after a rather confused procedure). He concludes that the problem was constructed from the observation that if a trapezium with widths $a = 1;40$, $b = 0;20$ (and "length" for instance 1, but this is unimportant) is cut into equally broad strips, then the area of the three "upper" strips equals that of the six "lower" ones. Even though the mathematics is better than in AO 17264, the similarity is at the very least suggestive.

If we now return to Problem 1, we may make two numerical observations. One is that the area of the lower partial trapezium is an aliquot part (namely $^1/_{10}$) of the area of the whole; the other is that its "length" is another aliquot part (namely $^1/_5$) of the "length" of the whole. The configuration dealt with in this problem may thus have resulted, either from experimentation similar to that behind Problem 4 or (see Figure 3) from something akin to that behind AO 17264 (*may* have resulted, need not by necessity; the presence of two numerical coincidences of which at least one is accidental serves to warn that both could be so—on the other hand aliquot parts *are* important, corresponding either to equal partition or to partition in arithmetical progression). Figure 3 (for convenience based on

[5] See the analysis in Høyrup, *Lengths, Widths, Surfaces*, pp. 244–249.

[6] As will be remembered, the Babylonian sexagesimal place value system was a floating-point system: the notation, originally intended for intermediate calculations but also used in the mathematical school texts, gave no indication of the absolute order of magnitude.

[7] See O. Neugebauer, *Mathematische Keilschrift-Texte* I (Berlin: Julius Springer, 1935), p. 133 for the mistaken order of magnitude. That the procedure involves the scaling to square shape was pointed out by Piedad Yuste in her dissertation "Modelos geoméricos de la matemáica babilónica: Pruebas y refutaciones"(Madrid: Dpto. de Lógica, Historia y Filosofíade la Ciencia de la Facultad de Filosofía, Universidad Nacional de Educación a Distancia (UNED), 2003).

[8] See the analysis in Lis Brack-Bernsen & Olaf Schmidt, "Bisectable Trapezia in Babylonian Mathematics". *Centaurus* **33** (1990), 1–38.

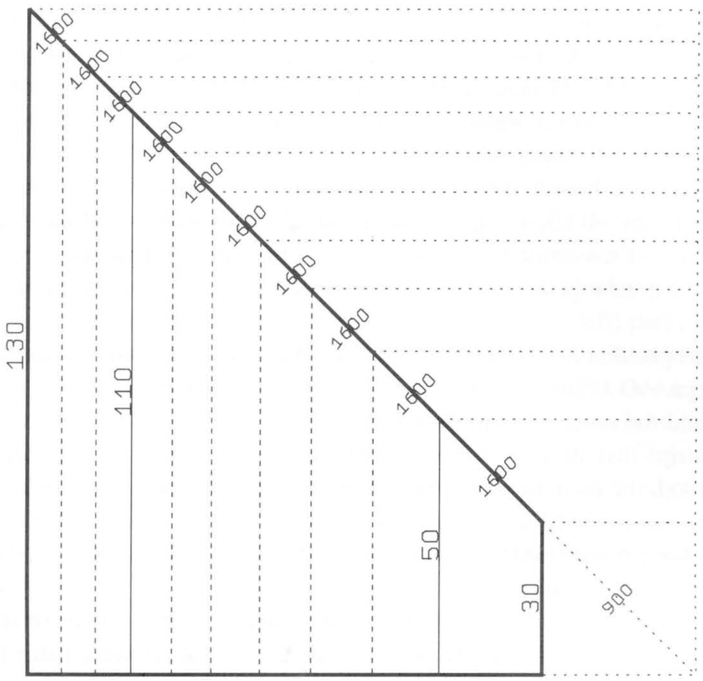

Figure 3.

the configuration inscribed in a square, but this is not essential) may be understood in two ways. A trapezium with widths $a = 2, 10$ $(= 130)$, $b = 30$ may have been divided into ten strips of equal area; then computation of the transversals (whose squares are found as in Problem 1, by repeated addition of 1600 (= 26,40) to the 900 (=15,0) that represents the area of the small square to the right) shows that those delimiting the first strip and the first 8 strips turn out to be integer. Alternatively, we might have started from a double trapezium or gnomon with widths 30 and 50 and continued to add gnomons with the same area (1600, in the square case), getting integer upper widths when eight gnomons are aggregated, and again with ten (at which point we stop); this could easily be established by repeated addition of 26,40 $(= 50^2 - 30^2)$ and inspection of a table of squares.

Referring to the construction behind problem 4, Muroi points out that the present text is "a very important source material for the study of Babylonian mathematics" by suggesting "how the problem itself was made by an ancient scribe". The obvious similarity to TMS 17264 strengthens this observation, by showing that several related texts presuppose variants of the same technique; the possible use of an analogous principle in the construction behind Problem 1 lends even further strength to it.

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